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Relativistic Quantum Mechanics of Fermions

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Abstract

We present a modification of the Dirac equation that allows us to formulate a relativistic quantum mechanics for spin-$\frac{1}{2}$ fermions in an external electromagnetic field, with a probabilistic interpretation similar to that in nonrelativistic quantum mechanics and based on an indefinite charge density. We find that stationary states cannot be interpreted in this manner, and we replace them by quasistationary states. We also include a general discussion of the difficulties and possible generalizations of this approach.

1. *Introduction*

Very soon after the formulation of nonrelativistic quantum mechanics, which is covariant under Galilean transformations, attempts were made to generalize it to a relativistic formulation, covariant under Lorentz transformations, which, after all, were known to be the more generally valid ones. The results are typically represented by the Klein-Gordon equation, which describes spin-0 particles, and the Dirac equation, for spin-$\frac{1}{2}$ particles.

Immediately a number of difficulties were discovered, connected with either or both these equations, which led to unsatisfactory interpretations and to the practical abandonment of relativistic quantum mechanics in favor of the theory of quantized fields.

The Klein–Gordon equation was originally rejected for two main reasons. A probability density could not be found that was positive and that gave rise to a conserved probability, and it was a second-order equation in time, which required a doubling of the amount of information needed to specify the state at a given time. Later on, it was reinterpreted (Pauli & Weisskopf, 1934; Feshbach & Villars, 1958) by changing the notion of probability density to that of a charge density, which could describe both a particle and its antiparticle, explaining, incidentally, the increase in the amount of information needed to determine a state. Nevertheless, no probabilistic interpretation along the lines of the one generally accepted in nonrelativistic quantum mechanics was suggested, until we recently

formulated one (Marx, 1969a) for a charged scalar particle in an external electromagnetic field. We based this interpretation on the use of the causal Green function or Feynman propagator for the Klein-Gordon equation, and the idea of Stueckelberg and Feynman, that an antiparticle is a particle 'going' backwards in time. We further showed (Marx, 1970a) how this theory could be extended to several identical particles, with the help of Dirac's many-time formalism. The resulting theory deals with a fixed number of particles, while allowing at the same time for pair creation and annihilation.

The Dirac equation initially replaced the Klein-Gordon equation because it yielded a positive probability density and was a first-order equation in time, although this feature was introduced at the cost of not just doubling but quadrupling the number of amplitudes. This was later successfully interpreted in terms of the spin of the electron and its antiparticle, the positron. But other severe difficulties in its interpretation were not satisfactorily explained, unless one accepts such awkward devices as the hole theory of the positron,† particularly unsuited in a one-particle formulation. On the other hand, our probabilistic interpretation could not be extended to the Dirac field precisely because it had a positive probability density instead of an indefinite charge density. We could obtain the desired type of charge density if we used the Klein-Gordon equation for four-component spinors (Marx, 1967) or two-component spinors (Marx, 1970c), but the former is not appropriate for electromagnetic interactions, which remain essentially unchanged (Marx, 1969a) while the number of amplitudes is doubled again to eight, and the latter requires the explicit introduction of the observer in the theory and changes somewhat the nature of the interaction. In this paper we propose another modification of the Dirac equation, in the form presented in Section 2, that leads to an indefinite charge density while introducing only a change of sign in the interaction, as shown in Section 3.

It is also open to question why one should make a sizable effort to find a consistent theory for a relativistic quantum mechanics, when these problems disappear in the usual formulation of quantum field theory. There are several reasons why we feel that this should be done. The vaguest and most general one is that there is a place in the hierarchy of physical theories for a relativistic quantum mechanics, and there is no reason to leave this area unexplored. A more technical reason refers to the difficulties which still beset quantum field theory in its various forms, in particular divergences and a general lack of either a firm mathematical background or practical computational procedures, and its limited success outside of quantum electrodynamics. We can restrict ourselves to the use of normalized states, and the fact that we are dealing with a fixed number of variables excludes the occurrence of certain divergences, related to closed fermion loops in quantum electrodynamics. Also, the quantization of fermion fields cannot

† See, for instance, Bjorken & Drell (1964).

be carried out quite according to the canonical procedures (Marx, 1969b), and anticommutators usually have to be introduced as an afterthought. Even in Fock space (Marx, 1970b), anticommutators are introduced most naturally in terms of particle amplitudes. Finally, the fact that fermion lines in Feynman graphs are never broken, a consequence of conservation of angular momentum, suggests that it might be possible to treat fermions as particles, coupled to classical or possibly quantized boson fields.

It is also customary to formulate eigenvalue problems in terms of the Klein-Gordon (Bethe & Jackiw, 1968) and Dirac (Bjocken & Drell, 1964) equations, and the eigenvalues of the latter are in good agreement with the energy levels of the hydrogen atom. But once again a probabilistic interpretation of the corresponding eigenfunctions is either not possible or unreasonable. We find that this continues to be the case in our formulation, and in Section 4 we present an alternative approach in terms of quasi-stationary states.

We finish this paper with a general discussion of problems and perspectives of our approach in Section 5.

We use natural units, the time-favoring metric in space-time, and the modified summation convention for repeated lower Greek indices. Other notation used is the same as in Marx (1968, 1969a).

2. *The Dirac Equation*

We take as our point of departure the Dirac equation for the wave function $\psi(x)$ for a particle of charge $-e$ in an external (given) electromagnetic field $A_\mu(x)$,

$$(-iD_\mu\gamma_\mu + m)\psi = 0 \tag{2.1}$$

where

$$D_\mu = \partial_\mu - ieA_\mu \tag{2.2}$$

The study of the position operator (Schröder, 1964) and the angular momentum (Marx, 1968) of the free Dirac field clearly indicates that the proper choice for probability amplitudes in momentum space is the usual set of $b_\lambda(\mathbf{p},t)$ and $d_\lambda(\mathbf{p},t)$ in equation (M-46),† which become operators in a quantum field theory. In configuration space we use $g(x)$, formed by the two-component spinors $g^{(+)}(x)$ and $g^{(-)}(x)$ defined by equations (M-56) and (M-57), which are independent of our choice of spin states. They are related to $\psi(x)$ by a Foldy–Wouthuysen transformation (Foldy & Wouthuysen, 1950; Bjocken & Drell, 1964)

$$\psi(x) = \left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2}\left(\frac{1+i\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m}\right)g(x) \tag{2.3}$$

† By equation (M-46) we mean equation (46) in Marx (1969a).

$$g(x) = \left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2}\left(1 - \frac{i\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m}\right)\psi(x) \tag{2.4}$$

where $\tilde{E}$ is the integral operator

$$\tilde{E} = +(-\boldsymbol{\nabla}^2 + m^2)^{1/2} \tag{2.5}$$

We keep the relation (2.3) or (2.4) between ψ and g even in the presence of an interaction. The approximate forms of the Foldy–Wouthuysen transformation in that case still seek to diagonalize the Hamiltonian, while we want to obtain a coupling between positive and negative frequency amplitudes. Equation (2.1) reduces to

$$i\dot{g}(x) = H' g(x) \tag{2.6}$$

where

$$\begin{aligned} H' = \gamma_0 \tilde{E} - e\left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2}\Bigg(A_0 - \frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m}A_0\frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} \\ + i\gamma_0\frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m}\boldsymbol{\Sigma}.\mathbf{A} + i\gamma_0\boldsymbol{\Sigma}.\mathbf{A}\frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} + iA_0\frac{\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m} \\ - i\frac{\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m}A_0 - \boldsymbol{\alpha}.\mathbf{A} - \frac{\boldsymbol{\alpha}.\boldsymbol{\nabla}}{\tilde{E}+m}\boldsymbol{\alpha}.\mathbf{A}\frac{\boldsymbol{\alpha}.\boldsymbol{\nabla}}{\tilde{E}+m}\Bigg)\left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2} \end{aligned} \tag{2.7}$$

We note that the above H' is Hermitian with respect to the scalar product

$$(g, g')' = \int d^3x g^\dagger g' \tag{2.8}$$

which implies that the (positive) quantity

$$N = \int d^3x \psi^\dagger \psi = \int d^3x g^\dagger g \tag{2.9}$$

is conserved. This leads to the interpretation of $\psi^\dagger\psi$ as a probability density, but we again (Marx, 1969a) note that it is not equal to $g^\dagger g$, and that neither leads to a consistent probabilistic interpretation of the wave function when the field is not free. In particular, when we set $N^{(+)}$, defined by

$$N^{(+)} = \int d^3x \rho^{(+)}(x), \qquad \rho^{(+)} = g^{(+)\dagger} g^{(+)} \tag{2.10}$$

equal to 1 initially, and $N^{(-)}$, defined by

$$N^{(-)} = \int d^3x \rho^{(-)}(x), \qquad \rho^{(-)} = g^{(-)\dagger} g^{(-)} \tag{2.11}$$

equal to 0 finally, equation (M-62) shows that $N^{(+)}$ is greater than 1 finally, and cannot be interpreted as a probability. If we use a retarded Green function instead of a causal one, we can set $N^{(+)}$ equal to 1 and $N^{(-)}$ equal to 0 initially; then both $N^{(+)}$ and $N^{(-)}$ are positive, add up to 1 finally, and could be interpreted as probabilities for particles and antiparticles. But the charge, which is the difference between the two, is clearly less than 1, violating one of the most basic conservation laws of physics.

3. *Modification of the Dirac Equation*

The above-mentioned difficulties are eliminated in the quantized theory of the Dirac field when the wave functions become anticommuting operators, and the charge operator is antisymmetrized or normal-ordered. The latter operation is well defined only for the free field, or for the field operators in the Dirac picture. Anticommuting creation operators also make the phase of a state vector dependent on the order in which these operators act on the vacuum to produce the state. These creation and annihilation operators closely correspond to the negative and positive frequency parts of the field and its Hermitian conjugate.

On the other hand, if we want to extend our probabilistic interpretation to the relativistic quantum mechanics of spin-$\frac{1}{2}$ particles based on the Dirac equation, we must modify it in such a way that the charge

$$Q = -e \int d^3x(\rho^{(+)} - \rho^{(-)}) \tag{3.1}$$

be a conserved quantity. This is the case when the Hamiltonian is a self-adjoint operator with respect to the indefinite metric

$$(g, g') = \int d^3x g^\dagger \gamma_0 g' \tag{3.2}$$

which H' given by equation (2.7) is not. In the notation used in connection with an indefinite metric, the new Hamiltonian H has to satisfy

$$H^\star = \gamma_0 H^\dagger \gamma_0 = H \tag{3.3}$$

where the star indicates the adjoint with respect to the metric (3.2), and the dagger indicates the Hermitian conjugate with respect to the positive metric (2.8). In submatrix notation, equation (3.3) becomes

$$\begin{pmatrix} H_{++}^\dagger & -H_{-+}^\dagger \\ -H_{+-}^\dagger & H_{--}^\dagger \end{pmatrix} = \begin{pmatrix} H_{++} & H_{+-} \\ H_{-+} & H_{--} \end{pmatrix} \tag{3.4}$$

We try to remain close to the successful results of quantum electrodynamics, and limit ourselves to a change of signs in the interaction Hamiltonian. It is easy to see that it is sufficient to replace either H'_{+-} or H'_{-+} by its negative in equation (2.7). It is irrelevant which one is chosen, since changing the signs of both off-diagonal terms is equivalent to changing the sign of either $g^{(+)}$ or $g^{(-)}$. We replace H'_{+-} and obtain

$$\begin{aligned} H = \gamma_0 \tilde{E} - e\left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2} \Bigg(& A_0 - \frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} A_0 \frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} \\ & + i\gamma_0 \frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} \boldsymbol{\Sigma}.\mathbf{A} + i\gamma_0 \boldsymbol{\Sigma}.\mathbf{A} \frac{\boldsymbol{\Sigma}.\boldsymbol{\nabla}}{\tilde{E}+m} - iA_0 \frac{\boldsymbol{\alpha}.\boldsymbol{\nabla}}{\tilde{E}+m} \\ & + i\frac{\boldsymbol{\alpha}.\boldsymbol{\nabla}}{\tilde{E}+m} A_0 + \boldsymbol{\gamma}.\mathbf{A} - \frac{\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m} \boldsymbol{\gamma}.\mathbf{A} \frac{\boldsymbol{\gamma}.\boldsymbol{\nabla}}{\tilde{E}+m} \Bigg) \left(\frac{\tilde{E}+m}{2\tilde{E}}\right)^{1/2} \end{aligned} \tag{3.5}$$

In this simple manner we have obtained a theory with an indefinite charge density and a conserved charge. We can thus extend the probabilistic interpretation of scattering in an external field for one or several identical particles, as presented by Marx (1969a, 1970a, 1970c), to spin-$\frac{1}{2}$ fermions that obey a slightly modified Dirac equation. The wave functions for several fermions have to be antisymmetric in all groups of variables, and the states of particles have to be specified at the initial time, while those of antiparticles must be given at the final time.

The form in which the equation has been cast does not lend itself to the determination of the transformation laws of g under restricted Lorentz transformations (other than ordinary rotations) and gauge transformations. Although we have started from a Lorentz and gauge invariant equation, we cannot easily ascertain the effects of the change of sign of H_{+-}. In any event, a Lorentz transformation should not mix $g^{(+)}$ and $g^{(-)}$, while charge conjugation should exchange them. We find that under the latter, $g(x)$ goes to†

$$g'(x) = i\alpha_2 g^*(x) \tag{3.6}$$

since

$$\alpha_2 \boldsymbol{\gamma}^* \alpha_2 = \boldsymbol{\gamma}, \qquad \alpha_2 \boldsymbol{\alpha}^* \alpha_2 = -\boldsymbol{\alpha}, \qquad \alpha_2 \boldsymbol{\Sigma}^* \alpha_2 = \boldsymbol{\Sigma} \tag{3.7}$$

$$-i\alpha_2 H^*(e)(i\alpha_2)^{-1} = H(-e) \tag{3.8}$$

where changing the sign of e is equivalent to replacing $A_\mu(x)$ by

$$A_\mu'(x) = -A_\mu(x) \tag{3.9}$$

In other words, when $g(x)$ is a solution of the Schrödinger equation with given fields A_μ, equation (3.6) provides a solution when all the components of the fields change sign. We also note that the helicity spinors obey

$$i\sigma_2 \chi^*_{-\lambda}(\hat{p}) = \lambda \chi_\lambda(\hat{p}) \tag{3.10}$$

Another important transformation is time reversal, which affects the fields and the coordinates. It is given by

$$\mathbf{x}' = \mathbf{x}, \qquad t' = -t \tag{3.11}$$

$$A_0'(x') = A_0(x), \qquad \mathbf{A}'(x') = -\mathbf{A}(x) \tag{3.12}$$

$$g'(x') = i\gamma_1 \gamma_3 g^*(x) \tag{3.13}$$

which is essentially‡ the choice in the usual Dirac theory, since we still have

$$i\gamma_1 \gamma_3 H^*(x)(i\gamma_1 \gamma_3)^{-1} = H(x') \tag{3.14}$$

This time reversal operation does not interchange particle and antiparticle amplitudes; it can be combined with charge conjugation to obtain the so-called strong time reflection.

† The change in the Hamiltonian brings about the replacement of $i\gamma_2$ by $i\alpha_2$ in equation (3.6), which has the somewhat puzzling effect of changing the sign of g under two successive charge conjugations. The asterisk indicates the complex conjugate.

‡ The only difference is the factor i, which is determined by the transformation properties of the charge conjugate solution.

4. *Quasistationary States*

One of the principal problems in nonrelativistic quantum mechanics is the determination of bound states. These are eigenstates of the Hamiltonian, which is independent of time, corresponding to negative eigenvalues and eigenfunctions that vanish at infinity. This has been formally extended to the relativistic equations (Bjocken & Drell, 1964; Bethe & Jackiw, 1968), but without attaching any consistent interpretation in terms of probability densities to the eigenfunctions. A simple fact that points out the difficulties† is that an electron can be bound in the field of a nucleus, while a positron cannot; if they are mixed in the wave function, it appears unnatural for a positron to be found near the nucleus.

Furthermore, the whole idea of a time-independent Hamiltonian and a central potential is ill-suited for a relativistic treatment; and is usually considered only as an approximation to the real situation. A more realistic treatment would involve not only a wave function for both particles, but also a dynamical (as opposed to external) electromagnetic field.

Our probabilistic interpretation is heavily dependent on the specification of initial and final conditions, and these have no place in a stationary state solution, which is essentially independent of time.

The above arguments lead to the conclusion that what we need is an approximation for stationary states that has the right nonrelativistic limit for bound states. We call them quasistationary states, and define them in the following manner for bound particles.

We assume that the Hamiltonian is independent of time, and we find eigenvalues E_n and eigenfunctions $g_n^{(+)}$ for the equation

$$H_{++}g_n^{(+)}(\mathbf{x}) = E_n g_n^{(+)}(\mathbf{x}) \tag{4.1}$$

We note that they do not lead to eigenvalues and eigenfunctions of the full Hamiltonian; if the off-diagonal terms do not vanish, there are no eigenfunctions with a vanishing $g^{(-)}$. We now assume that initially we have a particle in one such state, with the wave function normalized to 1, and that there is no antiparticle in the final state, and solve the problem in the usual manner. We obtain a certain amplitude for the antiparticle at the initial time, with the corresponding probability density and total probability, and the amplitude for the particle at the final time. A numerical calculation for a typical scattering problem (Walter & Marx, 1970a) leads us to believe that the probability for pair annihilation is small, and that it is most likely that the particle remains in the same state. In other words, if we have a large number of hydrogen atoms in a stationary state (the lack of a dynamical electromagnetic field makes all stationary states stable in nonrelativistic quantum mechanics), most of them will remain in that state, but a few will have their electrons annihilated by positrons. We also

† We discuss the problem of relativistic stationary states in more detail in a forthcoming paper with J. F. Walter. (Walter & Marx, 1970b.)

expect these energy levels to differ little from those calculated for the usual stationary states, since the contribution from the antiparticle amplitudes has to be small.

It is obvious that a similar argument produces eigenfunctions and eigenvalues for antiparticles when the sign of the potential is changed. The eigenvalues will be $-E_n$, and we have to remember that in a relativistic theory the energy levels are shifted by an amount equal to the rest energy; nonrelativistic negative energies correspond to

$$E_n - m < 0 \tag{4.2}$$

Actually, we expect these differences to be small in magnitude compared to the mass. It might be argued that an increase in the magnitude of the potential, or of the coupling constant e, would lead to large differences and probably negative values of the E_n themselves, mixing particle and antiparticle states as in the Klein paradox (Klein, 1929); in such a case, the entire formulation in terms of quasistationary states is not valid, which is not surprising for an approach that is an approximation in the first place.

A similar problem arises in the discussion of scattering states (Walter & Marx, 1970b), where

$$E - m > 0 \tag{4.3}$$

with the added complication that stationary states are not normalizable. Both $g^{(+)}$ and $g^{(-)}$ depend on time through $\exp(-iEt)$, which is at least unusual for a negative frequency part. We can again formulate a theory in terms of quasistationary states, but it is likely that the time-dependent approach is better suited for a scattering problem.

5. *General Remarks*

We have formulated, in this and other papers, a general framework for a relativistic quantum mechanics, both for bosons and for fermions. Many details remain to be worked out, and it is possible that some of the equations might have to be corrected, but we believe that the high degree of consistency indicates that our theory is basically correct.

So far we have refrained from giving a detailed discussion of the relationship between the equations and experiments carried out either in a laboratory or in thought. Very often these explanations only become possible after enough calculations have been carried out to enable one to fully understand the nature of the solutions. In this case, the basic problem is the specification of final conditions in a process. This is mathematically well defined, but it is contrary to our ideas of causality and the dynamical development of a system in time. This might not be a limitation of the theory or physics itself, but one due to our nature as observers made out of matter, rather than antimatter. If the universe is symmetric with respect to charge conjugation, our environment is certainly a large fluctuation; it is possible that

in case we ever observe a region in which antimatter is preponderant, we would notice that the Second Law of Thermodynamics and the 'arrow of time' are reversed.

Feynman has used the idea that physical processes are completely laid out in space-time, while we gradually advance along our time axis and observe them. It is also easier to consider the combination of particle scattering with pair annihilation, in which the particle state is determined initially while limiting ourselves to demand no antiparticles at the final time, than that of antiparticle scattering with pair creation, where the antiparticle state has to be specified at the final time. Our experience with antiparticles is still rather limited and it is not certain that we will ever be able to have large amounts of antimatter; instead of 'preparing' an antiparticle system in a final state, we might just select those experiments that correspond to predetermined final conditions. This distinction is really not well defined, especially when we consider scattering cross-sections, which are essentially probabilities to find a system in a certain final state for a given initial state, and which are the object of most high-energy experiments.

We have also consistently worked in the Schrödinger picture, which we find closer to our intuition of a physical process. In the Heisenberg picture, which is generally preferred when Lorentz invariance plays a role, the dynamics is assigned to the operators while the initial conditions determine a constant state vector; we have not developed a corresponding relativistic theory in which both initial and final conditions are specified. The preferred role of the time as a parameter in the Schrödinger picture makes manifest Lorentz covariance impossible, except as far as the replacement of the time axis by an arbitrary time-like direction is concerned. In that case we introduce a unit vector n that represents the state of motion of the observer, avoiding the unnecessary assumption that the observer is at rest in the reference frame. We thus separate the mathematical notion of a system of coordinates from the physical one of an observer. The use of the Schrödinger picture also serves to emphasize the conceptual difference between the time coordinate, that parametrizes the dynamical development of a system, and the space coordinates, which are a set of continuous indices for the generalized coordinates of the system. When we consider the dynamical problem of one particle without internal degrees of freedom in nonrelativistic quantum mechanics, we can specify a unit vector in the Hilbert space of functions of one three-vector variable at the initial time and we calculate the subsequent 'motion' of this vector in Hilbert space. Similarly, for the corresponding relativistic problem, we consider *two* such vectors simultaneously; we specify one of them, with norm 1, at the initial time, we demand that the other vanish at the final time, and we proceed to compute both at all intermediate times. The norms of these vectors are no longer 1, and they give the probability of finding the 'particle' in a particle state or in an antiparticle state at a given time. Operators other than the Hamiltonian have been defined mostly in terms of particle and antiparticle amplitudes separately. For instance, there is a position operator for the particle and

one for the antiparticle, and it does not appear realistic in terms of the present discussion to compute a combined expectation value, using the indefinite metric. Strange concepts, such as *Zitterbewegung*, are obtained when we overlook this point, as well as when we make the wrong choice of probability amplitudes in configuration space. Most formulations are usually carried out in momentum space, where the right amplitudes are easier to determine; this choice is important both for relativistic quantum mechanics (Marx, 1969a) and for the theory of quantized boson fields (Marx, 1969b). It is also plausible that we might be able to restrict the definition of most operators to free particles, since measurements are normally carried out outside the region of interaction for the process under study.

One practical difficulty of great importance is the lack of familiar procedures to deal with operators defined in terms of $\tilde{E}$; in particular, we know of no simple expression for this operation carried out on a product of functions. As a consequence, we have not been able to find a current density that conservation of charge should associate with the charge density, if we accept that the latter is the difference between the probability densities times e (a nontrivial assumption). Schröder (1964) distinguishes between the position of the particle and the position of the charge, a distinction we prefer to avoid. We do not know, for instance, how to find the energy levels for the quasistationary states, but we expect them to be very close to those of stationary states, which are well known and agree with experiment. It is a matter of conjecture whether there is any relation between such corrections and the Lamb shift.

In the particular case of spin-$\frac{1}{2}$ fermions, the questions about Lorentz and gauge invariance are especially difficult to answer, both for the present theory and that by Marx (1970c); we should point out that, in the latter case, the spectrum for the stationary states of a particle in a Coulomb field is the same as that of scalar particles, in disagreement with experimental results for the hydrogen atom. It is possible that a different modification of the Dirac equation would also have the required property of leading to an indefinite charge density, while having advantages in other respects. Nevertheless, we feel that the success of the calculations in quantum electrodynamics makes the need of broad changes unlikely.

A limitation of both nonrelativistic and relativistic quantum mechanics is the lack of an interaction with the electromagnetic field, sometimes avoided by such devices as the semiclassical theory of radiation. It is straightforward to include (Marx, 1970c) a term representing the free electromagnetic field, when we start from a Lagrangian density; we are led to a problem in the theory of classical fields, which can be interpreted in terms of probability amplitudes. The unorthodox specification of initial and final conditions, though, leads to the need of introducing gauge independent fields (Marx, 1970c; Goldberg & Marx, 1968). Also, the equations are no longer linear, due to the interaction term, with the obvious consequences for the superposition principle. But when this approach is

used (Bjocken & Drell, 1964) in combination with a perturbation expansion, we can at least formally obtain the familiar results in quantum electrodynamics.

We should also point out the role the observer plays in the determination of the gauge independent part of the electromagnetic potential. This is essentially a nonlocal procedure, and depends on the choice of hyperplanes, perpendicular to n; it is similar to the problem that arises in the separation of the radiation field from the velocity (Coulomb) field in classical electrodynamics.

Another open question is the need for, and the feasibility of, the quantization of these boson fields. Bosons are essentially their own antiparticles, when we consider simultaneously groups of bosons, such as the three pions, as represented by a real field with three components in isospin space. If we consider the classical field as a system with an infinite number of degrees of freedom, we once more have to resort to an observer to define the state of the system (at a given time), and it is this classical generalized coordinate which becomes an operator when a canonical quantization procedure is used. We would also probably have to carry out this quantization in the Schrödinger picture.

Alternatives to dynamical boson fields, such as various forms of action-at-a-distance theories, do not appear very promising at this time.

It should be remembered that a real field does not lend itself to the use of a causal Green function, but we have to choose either the retarded or the advanced one, presumably depending on the specification of the boundary conditions for the particles. This would introduce further changes in the results of calculations of Feynman graphs in quantum electrodynamics.

Some results that might seem strange at first sight can be explained with due care. For instance, we find (Marx, 1970c) that a charged particle initially at rest with no incident radiation field does emit radiation while building up an antiparticle amplitude at the initial time; we can interpret this result by saying that a particle at rest has a finite probability of being annihilated by an antiparticle with the emission of radiation.

Further work is needed to make sure that we have a consistent relativistic quantum mechanics, and to ascertain whether we can extend it to obtain the results of quantum electrodynamics. If this program is successful, it appears likely that it could be further extended to include strong and weak interactions of particles and fields.

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